Period Doubling Route to Chaos

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A simple nonlinear physical system, the driven diode resonator, demonstrates the period doubling route to chaos. Harmonics, subharmonics, and ultraharmonics have been identified on the Fourier transform of the diode voltage, and modeling of the circuit in P-Spice replicates the experimental data.

Chaos can be defined as a system that diverges exponentially from initial positions varying by a small degree[1]. A driven diode resonator is such that exhibits these characteristics. Depending on the frequency or amplitude of the input, the output may be either periodic or chaotic.

The Circuit:

Chaos is exhibited in the diode resonator circuit due to the unrecombined charges that cross the p-n junction when the diode is in the forward bias mode. When the diode switches biases, the charges diffuse back to their original plate making the diode act as a capacitor. The larger the forward current, the greater the amount of charges that cross the junction and the longer the system will need to return to its reverse bias equilibrium. If the reverse current is unable to reach equilibrium before the forward bias, then the next cycle depends upon the previous cycle. This may lead to different parameters for the beginning of each cycle[1]. Under such conditions, the system may become chaotic.

One of the routes to chaos is by period doubling. In this case, the period continues to double until there are no more stable states available. When driven at a frequency near the diode's resonant frequency, the circuit can exhibit periodic behavior. As the driving amplitude is increased, the periodic state becomes unstable. The state divides into two frequencies dependent on the resonance. The harmonic frequency remains but a second frequency appears at half the harmonic. This is defined as period doubling. Further increase in the amplitude results in the splitting of the two periods, giving quadrupling, octupling, and finally chaos[1].

A visual representation of this process can be seen in Fig. 2B. Although not drawn to scale, you can see in this diagram how the frequency splits repeatedly and with a set pattern. Periodicity splits into period doubling, which then splits into period quadrupling, and so forth. Hence we see that the system endures more and more period bifurcations. This will continue until the separation between the neighboring frequencies becomes indistinguishable. At this point, the system becomes unstable and chaotic. Further increase of the voltage will bring the system back to a linear state.

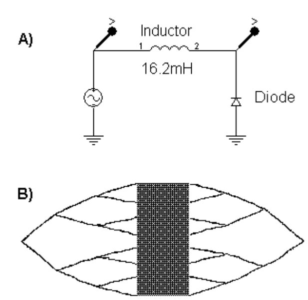

Fig. 1: A) Description of the diode resonator circuit. Composed of an inductor and diode in series. The voltage is measured across the inductor. B) Bifurcation diagram for the circuit. As the system approaches chaos, the periodic state will divide to a period-2 state, period-4 state, etc., until the adjoining states are too close to distinguish.

A Power Spectrum provides an easy way to identify a chaotic system[2]. This analysis shows us the power vs. frequency for a given system. In a periodic system, only one harmonic peak occurs, which is associated with the driving frequency. As the system progresses towards chaos, more peaks will occur, associated with the harmonics, subharmonics, and ultraharmonics of the system. For a truly chaotic system, there should be a spectrum of frequencies rather than specific peaks. However, multiple chaos may also occur, in which there is a broadening of the spectrum near certain frequencies.

A schematic of our circuit is found in Fig. 1A. The inductor and the diode determine the resonant frequency of the circuit, which is necessary to consider a driving frequency. This frequency was measured as 143 kHz and held constant throughout the experiment. The inductor was measured as 16.2 mH, using a notch filter circuit as described in Horowitz and Hill[3], and the internal capacitance of the diode was measured as 3 nF.

Based upon an experiment found in Enns[2], we wished to change one parameter and seek chaos. The variant parameter in this experiment was the amplitude of the input voltage. As the input voltage was increased, period bifurcations became visible. The voltage across the diode and the input voltage were measured with an oscilloscope and LabView. Input voltages were chosen so as to produce a period-1, period-2, period-4, period-8, and chaotic output. Fig. 1B shows the bifurcation diagram for the behavior of the system. The system will exhibit up to period-8 before chaos will appear.

Discussion:

Fig. 2A shows the Power Spectrum of the system when exhibiting periodic behavior. The harmonic frequency appears at that of the driving force, 143 kHz, while a small ultrasubharmonic appears at 286 kHz. Ultrasubharmonic frequencies are found at mf_0/n where n=1 and m=1,2,3,…[2]. Due to the sheer number of data points needed to create a power spectrum, 10 scope traces were recorded and combined.

Fig. 2B is the time series for the periodic solution. This is the raw data the computer received from the circuit. This display shows that the function of the circuit is similar to a sin wave and therefore periodic.

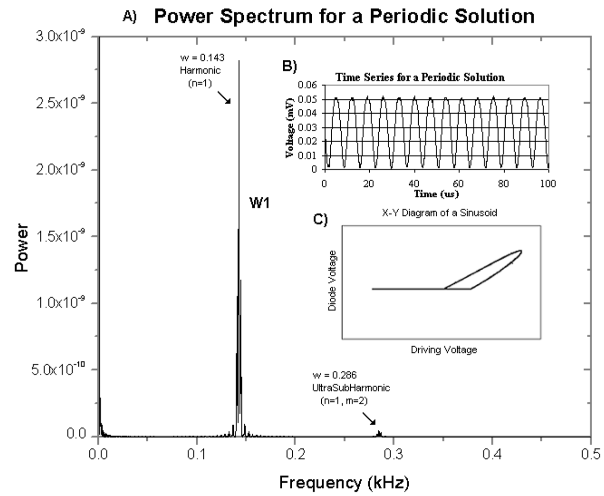

Fig. 2: Response of the driven diode resonator when exhibiting periodic behavior. Driven at a frequency of 143 kHz and amplitude of 50 mV. The harmonic frequency is at 143 kHz and the ultraharmonic is at 286 kHz (*f/2).* A) Power Spectrum; B) Raw data of the diode voltage; C) Phase plot of diode voltage vs. drive voltage[4].

Fig. 2C is the X-Y diagram for the periodic solution. This graph demonstrates the relationship between the input and output voltages of the system. From this diagram it is visible that there is only one frequency in this system.

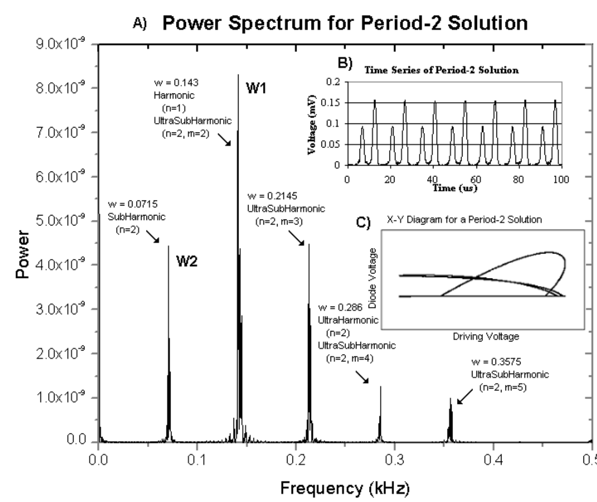

Fig. 3: Response of the driven diode resonator when exhibiting a period-2 solution. Driven at a frequency of 143 kHz and a voltage of 178 mV. The harmonic frequency of the system is at 143 kHz and the subharmonic frequency is 71.5 kHz ($f/2$). A) Power Spectrum; B) Raw data of the diode voltage; C) Phase Plot of diode voltage vs. drive voltage[4].

When the driving voltage is increased to 178 mV, the circuit exhibits its first bifurcation. The system has split into two frequencies, one of which is the harmonic frequency of the driving force, 143 kHz, and the other is half the harmonic frequency at 71.5 kHz (Fig. 3A). This second frequency is referred to as the subharmonic frequency. Subharmonic frequencies are found by f_0/n , where n = 1, 2, 4, 8, …[2]

This system exhibits the same ultraharmonic frequencies as seen in the periodic solution, but now there are ultrasubharmonic frequencies as well. The ultrasubharmonic frequencies can be found by mf_0/n, where n = 1, 2, 4, …., and m = 1, 2, 3….[2]

Fig. 3B exhibits the time series graph of the period-2 solution. The two different frequencies are clearly visible in this display. The small peaks are half way between the large peaks and about half the height.

Fig. 3C displays the X-Y diagram for the period-2 solution. The large broad peak is the harmonic frequency, whereas the thin, small peak is the subharmonic frequency.

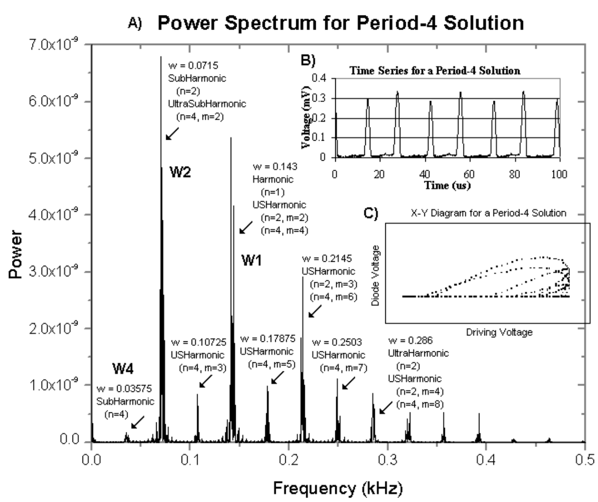

Fig. 4: Response of the driven diode resonator when exhibiting a period-4 solution. Driven at a frequency of 143 kHz and a voltage of 220 mV. The harmonic frequency of the system is 143 kHz and the subharmonic frequencies are at 71.5 kHz ($f_0/2$) and 35.75 kHz ($f_0/4$).A) Power Spectrum; B) Raw data of the diode voltage; C) Phase Plot of diode voltage vs. drive voltage[4].

When the driving voltage is increased to 220 mV, the circuit exhibits a second bifurcation into period quadrupling. In period quadrupling, they system has three stable frequencies. The harmonic frequency of the system remains at 143 kHz, but there are now two subharmonic frequencies found at 71.5 kHz and 35.75 kHz (Fig. 4A). The large increase in the peak of the second subharmonic is due to the addition of the ultrasubharmonic frequency of n=4.

Fig. 4B is the time series graph of the period-4 solution. In this system, there are four distinct peaks for the four frequencies of the system. The two small peaks are difficult to see in this scale.

Fig. 4C displays the X-Y diagram for the period-4 solution. This graph depicts the differences between the individual frequencies and amplitudes of the system.

Fig. 5 shows us the final bifurcation prior to chaos. The amplitude range of the period-8 solution is very small, and it took a few trials to obtain this solution. Again we see the harmonic at 143 kHz, and the second subharmonic as the largest peak. The increase of the ultraharmonic is due to its splitting.

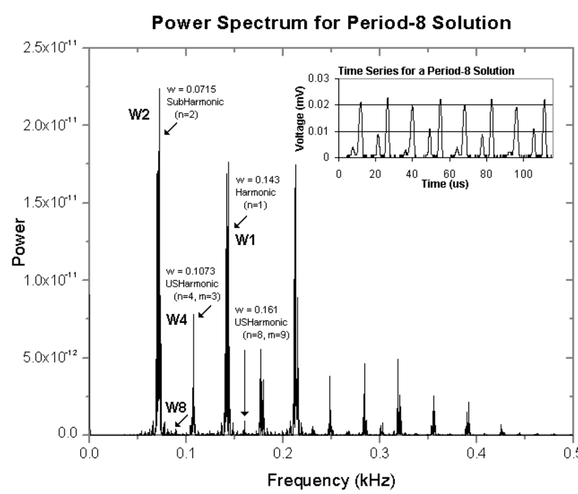

Fig. 5: Response of the driven diode resonator when exhibiting a period-8 solution. Driven at a frequency of 143 kHz and amplitude of 238 mV. A) Power Spectrum; B) Raw data of the diode voltage; C) Phase Plot of diode voltage vs. drive voltage[4].

Due to the fact that the period-8 solution is very hard obtain, the peak of the subharmonic for the n = 8 state is very small. This makes visibility at any amplitude difficult.

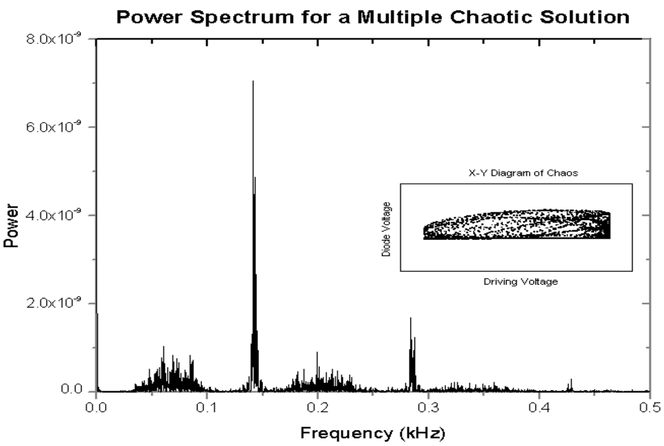

Fig. 6: Response of the driven diode resonator when exhibiting multiple chaotic behavior. Driven at a frequency of 143 kHz and amplitude of 304 mV. Peak frequencies are still visible at the resonant frequency 143 kHz but a wide spectrum of frequencies is seen between the harmonics**.**

Fig. 6 exhibits the chaotic behavior of the system. We can see that this is multiple chaos due to the broadening of the frequency spectrum around certain frequencies. The resonant frequency is still visible at 143 kHz. As we can see from the X-Y plot[4], there is no one frequency associated with the driving frequency, and thus the plot takes up all of phase space. If we were to watch the phase plot evolve, there would be pattern to the evolution of the phase plot.

The Model:

In order to validate our results, the above circuit was simulated using a program called P-Spice. The equation for the circuit[5] is:

$$L\frac{dI}{dt} = V_{in} - V_{Diode}$$

$$C\frac{dV_{Diode}}{dt} = \begin{cases} \text{I, when the diode is conducting} \\ \text{0, when not} \end{cases}$$

By inserting the same circuit diagram shown in fig. 1A, and varying the voltage amplitude of the input, we were able to simulate the following results:

The harmonic frequency obtained from the simulation is once again the same as the frequency observed from the experimental results. The subharmonic frequency of this system differs from the experimental frequency by 0.2%. The difference between the two systems is within experimental error. Our simulation of the experiment is valid.

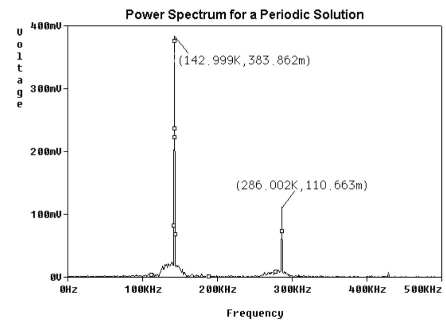

Fig. 7: Computer simulation of an FFT for a periodic solution to the diode resonator circuit. Resonant frequency is found at 143 kHz and an ultraharmonic frequency can be seen at 286 kHz.

The harmonic frequency obtained from the simulation is the same as the harmonic frequency observed from the experimental results. The ultraharmonic frequency of the simulation is also equal to the ultraharmonic frequency of the experimental results. These two figures lead us to believe that our findings were accurate.

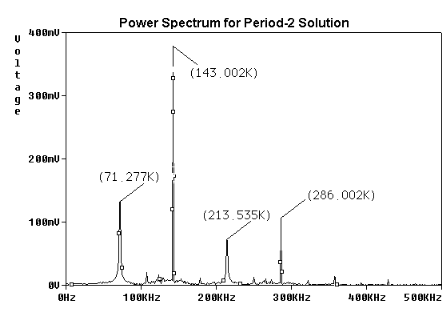

Fig. 8: Computer simulation of an FFT for a period-2 solution to a diode resonator circuit. The resonant frequency is found at 143 kHz and the subharmonic frequency can be seen at 71.3 kHz.

Conclusions:

The change in the amplitude of our driving frequency had a tremendous effect upon the outcome of our system. A mere increase of 200 mV progressed the system from periodic to chaotic. We also saw that this progression was predictable, as the bifurcation diagram suggested. We were able to identify the harmonics, subharmonics, and ultraharmonics in accordance to prediction.

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[1] Hunt, E., G. Johnson. (1993) "Keeping chaos at bay," IEEE Spectrum, Nov. 93, 32-36.

[2] R. Enns and G. McGuire. *Nonlinear Physics with Mathematica for Scientists and Engineers*. Birhkäuser, Boston: 2001.

[3] P. Horowitz and W. Hill. (1989) The Art of Electronics. 2nd ed. Cambridge University Press, Cambridge.

[4] R. Lua. "Period-doubling in a Simple Diode Circuit," University of Minnesota, http://www.physics.umn.edu/~rlua/chaos/, (28, Feb. 2002). This is a simulation of our circuit.

[5] K. Briggs. (1987) "Simple experiments in chaotic dynamics," Am. J. Phys. **55**, 1083-1089.